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ON A LINEAR DIFFERENTIAL GAME OF EVASION

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For a linear controlled system we examine the evasion problem on an infinite semi-interval of time. The paper abuts the investigations in [1 – 5]. The solution is effected by the scheme of control with a leader [3, 4].

1. We examine a controlled system described by the vector differential equation

$$dx / dt = Ax + Bu + Cv, \quad u \in P, \quad v \in Q^\alpha \tag{1.1}$$

Here x is the k-dimensional phase coordinate vector, u and v are $r^{(1)}$- and $r^{(2)}$-dimensional vectors, respectively; A, B, C are matrices with constant coefficients of dimension $k \times k$, $k \times r^{(1)}$, $k \times r^{(2)}$, respectively; the first and second player's controls are constrained by the conditions indicated above, where P and Q are convex compacta in the corresponding vector spaces. The symbol Q^α denotes the closed Euclidean α-

neighborhood of set Q, thus: $Q^\alpha = \{v = q + m\colon q \in Q, \|m\| \leqslant \alpha\}$. Here and subsequently, $\|m\|$ is the Euclidean norm of vector m. In the space $\{t, x\}$ we are given a certain set M, being a convex compactum in space $\{x\}$. The problem is to construct a strategy V ensuring, for all motions $x_\Delta[t]$ generated by this strategy, evasion from the ε-neighborhood M^ε of set M during the infinite semi-interval $t_0 \leqslant t < \infty$ for any action of the first player, constrained by the condition $u \in P$.

The terms encountered in this paper, e. g. strategies, motions, Euler polygonal lines, and their notation, are to be understood in the same sense as they were defined in [3].

Let us consider an auxiliary system, described by the vector differential equation

$$dw / dt = Aw + Bu_* + Cv_*, \qquad u_* \in P^\alpha, \qquad v_* \in Q \tag{1.2}$$

where the vectors w, u_*, v_* are of the same dimension as x, u, v, respectively. In the space $\{t, w\}$ we construct a set H consisting of points satisfying the condition $\rho(\{t, x[t]\}, M) \geqslant \varepsilon_0 > 0$ for $t \geqslant t_0$. Then, in accordance with the results in [5], the following alternative holds for the motions $w[t]$. One of the two conclusions is valid for every initial position $\{t_0, w_0\}$: either we can find an istant $\vartheta \geqslant t_0$ and a strategy $U_* \div u_* \times (t, w, v_*)$ (*) such that each motion $w[t, t_0, w_0, U_*]$ leaves H at least once for $t \in [t_0, \vartheta]$ or we can construct a strategy V_*° which guarantees the retention of every motion $w[t, t_0, w_0, V_*^\circ]$ in H for all $t \geqslant t_0$.

We shall assume that the second one of these constructions is fulfilled. In this case, the results of Sect. 2 of [5], in the terminology adopted in [3], signify that there exists a set $W \subset H$ which is a v-stable bridge $W_{\varepsilon_0}^\infty$. The symbol $W_{\varepsilon_0}^\infty$ denotes that this bridge does not intersect the set M^{ε_0} on the whole semi-axis $[t_0, \infty)$. Here, by the property of v-stability of the bridge $W_{\varepsilon_0}^\infty$ we mean the following. Suppose we have the position $\{t_*, w_*\} \in W_{\varepsilon_0}^\infty$. We select any $t^* > t_*$ and $u^*[t] \in P^\alpha$ arbitrarily measurable on the interval $[t_*, t^*]$. Then we can choose a measurable control $v^*[t] \in Q$ such that the motion $w[t]$ described by the equation

$$dw / dt = Aw[t] + Bu^*[t] + Cv^*[t]$$

remains on $W_{\varepsilon_0}^\infty$ on the interval $[t_*, t^*]$.

In [5] it was shown that it is possible to construct the second player's position strategy causing the motion $w[t]$ to evade set M for $t_0 \leqslant t < \infty$. Here we have noted that to realize the evasion for all the Euler polygonal lines $w_\Delta[t]$ approximating motion $w[t]$ requires additional stability conditions. The present paper is devoted to solving the problem of realizing such stability.

We construct the second player's strategy causing the approximating Euler polygonal lines $x_\Delta[t]$ to evade the set M^ε during the infinite semi-interval of time, with the aid of a control with leader $w[t]$ [3, 4]. The problem is formulated precisely in the following way. For a given initial position $\{t_0, x_0\}$ in the controlled system (1.1), find the strategy

$$V \div \{v(\tau, x, w), \qquad u_*(\tau, x, w), \qquad v_*(t, \tau, x, w, u_*(\cdot))\} \tag{1.3}$$

which for a sufficiently small partition step $\delta = \sup_i(\tau_{i+1} - \tau_i)$ $(i = 0, 1, \ldots)$ of the t-axis ensures the evasion of all the approximating Euler polygonal lines $x_\Delta[t] = x_\Delta[t,$

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*) Editor's Note : The symbol $\div$ (used throughout this paper), denotes the correspondence between the strategy and the function prescribing this strategy.

$t_0, x_0, V, u(\cdot)]$ from the ε-neighborhood M^ε of set M for all $t_0 \leqslant t < \infty$, i.e. for a sufficiently small $\varepsilon > 0$ we can find $\delta_0 > 0$ such that for all $\delta \leqslant \delta_0$ the strategy found guarantees the evasion of $x_\Delta[t]$ from M^ε for all $t \geqslant t_0$.

The scheme for constructing such a strategy is related to the solving of the problem on stabilizing a system described by the vector differential equation (s is the k-dimensional phase vector; l and m are the control vectors)

$$ds / dt = As - Bl + Cm \tag{1.4}$$

2. Let us describe the construction of motions $x_\Delta[t]$ and $w_\Delta[t]$. According to the problem statement, in the actual system (1.1) the control u is prescribed by the first player, and control v by the second player. In the auxiliary system (1.2) both controls u_* and v_* are prescribed by the second player. Then the second player is faced with the problem: by dealing with the controls u_*, v_* in system (1.2) and with the control v in system (1.1), to hold the motion $w_\Delta[t]$ on the bridge $W_{\varepsilon_0}^\infty$ (which is possible by virtue of the v-stability of bridge $W_{\varepsilon_0}^\infty$) and to manage things so that the motion $x_\Delta[t]$ of the actual system (1.1) traces out the motion $w_\Delta[t]$ of the auxiliary system (1.2). Then, using the terminology of the theory of stability of motion, the motion $x_\Delta[t]$ can be considered as the perturbed motion relative to the unperturbed motion $w_\Delta[t]$.

In accordance with the problem statement we examine two motions: the led [driven] motion $x_\Delta[t]$ in the given actual controlled system, described according to [3, 4] by the equation

$$dx_\Delta[t] / dt = Ax_\Delta[t] + Bu[t] + Cv(\tau_i, x_\Delta[\tau_i], w_\Delta[\tau_i]), \; x_\Delta[t_0] = x_0 \tag{2.1}$$

$$(\tau_i \leqslant t < \tau_{i+1}, \; i = 0, 1, \ldots)$$

and the leading [driving] motion $w_\Delta[t]$ produced by the auxiliary system and described by the equation (2.2)

$$dw_\Delta[t] / dt = Aw_\Delta[t] + Bu_*(\tau_i, x_\Delta[\tau_i], w_\Delta[\tau_i]) + Cv_*(t, \tau_i, x_\Delta[\tau_i], w_\Delta[\tau_i], u_*(\cdot)) \; (\tau_i \leqslant t < \tau_{i+1}, \; i = 0, 1, \ldots)$$

The controls u_*, v_*, v are chosen in the following manner. We first solve the problem of stabilizing system (1.4), i.e. find controls $l(s)$ and $m(s)$ which ensure the asymptotic stability of the trivial solution of system (1.4) with $l = l(s)$, $m = m(s)$. If system (1.4) is stabilizable (and this we assume), then the controls stabilizing system (1.4) exist and are the linear vector-valued functions $l = l(s)$ and $m = m(s)$ see [6]). Substituting the $l = l(s)$ and $m = m(s)$ found into (1.4), we obtain a linear asymptotically-stable system. Given the negative-definite quadratic form $\omega(s) = -\|s\|^2$, let us find a positive-definite quadratic form $L(s)$ for which the equality

$$(dL / dt)_{(1.4)} = (\partial L / \partial s)' (As - Bl(s) + Cm(s)) = -\|s\|^2 \tag{2.3}$$

is fulfilled. Here the symbol $(dL / dt)_{(1.4)}$ denotes the total time derivative relative to system (1.4), while the prime denotes transposition.

We shall form the motions $x_\Delta[t]$ and $w_\Delta[t]$ described by Eqs. (2.1) and (2.2) as follows. At the initial instant $t = t_0 = \tau_0$ we set $w_\Delta[t_0] = x_\Delta[t_0] = x_0$ and we arbitrarily select a control $v[t] = v[\tau_0] \in Q^\alpha$ on the semi-interval $[\tau_0, \tau_1)$. Also arbitrarily we select the control $u_*[t] = u_*[\tau_0] \in P^\alpha$ for $t \in [\tau_0, \tau_1)$ and we define $v_*[t] \in Q$ for $t \in [\tau_0, \tau_1)$ as a program control such that the condition $\{\tau_1, w_\Delta[\tau_1]\} \in W_{\varepsilon_0}^\infty$ is fulfilled for the motion $w_\Delta[t]$. The possibility of such a choice of control follows from the v-stability condition for bridge $W_{\varepsilon_0}^\infty$. Now suppose that at the instant $t = \tau_i$ ($i = 1, 2, \ldots$) we have realized the points $\{\tau_i, x_\Delta[\tau_i]\}$ and $\{\tau_i, w_\Delta[\tau_i]\}$. We construct the vector $s = x_\Delta[\tau_i] - w_\Delta[\tau_i]$ and we set up the equations of perturbed motion on the semi-interval $[\tau_i, \tau_{i+1})$ in the formalization adopted. We obtain

$$ds_\Delta[t] / dt = As_\Delta[t] + B(u[t] - u_{*i}) + C(v_i - v_{*i}) \quad (\tau_i \leqslant t < \tau_{i+1}) \tag{2.4}$$

At the instant $t = \tau_i$, from the values $x_\Delta[\tau_i]$, $w_\Delta[\tau_i]$, $s_\Delta[\tau_i]$ we construct the controls u_{*i}, v_i, $v_{*i}[t]$ in the following way. We construct the control $u_*[t] = u_{*i} = u_*(\tau_i, x_\Delta[\tau_i], w_\Delta[\tau_i])$ for $t \in [\tau_i, \tau_{i+1})$ as the sum

$$u_{*i} = p(\tau_i, x_\Delta[\tau_i], w_\Delta[\tau_i]) + l(s_\Delta[\tau_i]) \tag{2.5}$$

Here the function $l(s)$ is chosen from the solution of the stabilization problem for system (1.4), while the control p is selected from the maximum condition

$$\max_{p \in P} (\partial L / \partial s)'_{\tau_i} Bp(\tau_i, x_\Delta[\tau_i], w_\Delta[\tau_i]) \tag{2.6}$$

From the u_{*i} obtained we find $v_*[t, \tau_i, x_\Delta[\tau_i], w_\Delta[\tau_i], u_{*i}(\cdot)]$ as the program control $v_{*i} = v_{*i}[t]$ $(\tau_i \leqslant t < \tau_{i+1})$ so that the motion $w_\Delta[t]$, described by Eq. (2.2) is held on the bridge $W_{\varepsilon_0}^\infty$ for $\tau_i \leqslant t < \tau_{i+1}$

We construct the control $v[t] = v_i = v(\tau_i, x_\Delta[\tau_i], w_\Delta[\tau_i])$ for $t \in [\tau_i, \tau_{i+1})$ as

$$v_i = q(\tau_i, x_\Delta[\tau_i], w_\Delta[\tau_i]) + m(s_\Delta[\tau_i]) \tag{2.7}$$

Here the function $m(s)$ is chosen from the solution of the stabilization problem for system (1.4), while the control q is selected from the minimum condition

$$\min_{q \in Q} (\partial L / \partial s)'_{\tau_i} Cq(\tau_i, x_\Delta[\tau_i], w_\Delta[\tau_i]) \tag{2.8}$$

Equalities (2.5), (2.7) and the rule for constructing v_{*i} determine the strategy (1.3) to be constructed. This strategy solves the problem posed. In fact, the total derivative of quadratic form $L(s)$ by virtue of (2.4) on the semi-interval $\tau_i \leqslant t < \tau_{i+1}$ has the form

$$\begin{gathered} dL / dt = (\partial L / \partial s)_t' \, \theta, \; \theta = As_\Delta[t] + B(u[t] - p(\tau_i, x_\Delta[\tau_i], w_\Delta[\tau_i]) - \\ l(s_\Delta[\tau_i]) + C(q(\tau_i, x_\Delta[\tau_i], w_\Delta[\tau_i]) + m(s_\Delta[\tau_i]) - v_{*i}[t]) \end{gathered} \tag{2.9}$$

For convenience we rewrite (2.9) as

$$\begin{gathered} dL / dt = [(\partial L / \partial s)_t - (\partial L / \partial s)_{\tau_i}]' \, \theta + (\partial L / \partial s)_{\tau_i}' A(s_\Delta[t] - s_\Delta[\tau_i]) + \\ (\partial L / \partial s)_{\tau_i}' B(u[t] - p(\tau_i, x_\Delta[\tau_i], w_\Delta[\tau_i])) + \\ (\partial L / \partial s)_{\tau_i}' C(q(\tau_i, x_\Delta[\tau_i], w_\Delta[\tau_i]) - v_{*i}[t]) + \\ (\partial L / \partial s)_{\tau_i}' [As_\Delta[\tau_i] - Bl(s_\Delta[\tau_i]) + Cm(s_\Delta[\tau_i])] \end{gathered} \tag{2.10}$$

Taking (2.3), (2.6) and (2.8) into account we obtain the estimate

$$\begin{gathered} dL / dt \leqslant - \| s_\Delta[\tau_i] \|^2 + [(\partial L / \partial s)_t - (\partial L / \partial s)_{\tau_i}'] \theta + (\partial L / \partial s)_{\tau_i}' A \times \\ (s_\Delta[t] - s_\Delta[\tau_i]) \end{gathered} \tag{2.11}$$

Either continuous functions or bounded quantities occur in the right-hand side of inequality (2.11); therefore, the estimate

$$dL / dt \leqslant - \| s_\Delta[t] \|^2 + \gamma\delta, \quad \gamma > 0 \text{ is a constant} \tag{2.12}$$

is valid almost everywhere on the semi-interval $[\tau_i, \tau_{i+1})$ for (2.11).

In the space $\{s\}$ we now construct the β-sphere $\|s\| \leqslant \beta$ satisfying the following conditions: $\|l(s)\| \leqslant \alpha$, $\|m(s)\| \leqslant \alpha$ hold inside the β-sphere and, in addition, $\beta \leqslant \varepsilon_0$. Let us consider the surface $L(s) = c_1$, where $c_1 = \min(c_1', c_1'')$. Here the constant c_1' is chosen from the condition that the surface $L(s) = c_1'$ lies wholly inside the β-sphere, while the constant c_1'' is such that the surface $L(s) = c_1''$ is inscribed in the sphere $\|s\| \leqslant c_2$, i.e. from the condition $L(s) \leqslant c_1''$ follows $\|s\| \leqslant c_2$, where $c_2 = \varepsilon_0 - \varepsilon$. Obviously, we can find a number $\delta_0 > 0$ such that the sphere $\|s\|^2 \leqslant \gamma\delta_0$ lies within the surface $L(s) = c_1$. From inequality (2.12) it follows that the sign of the derivative $(dL / dt)_{(2.4)}$ is negative between the surfaces $\|s\|^2 = \gamma\delta_0$ and $L(s) = c_1$. This signifies that the motion $s_\Delta[t]$, starting from the sphere $\|s\|^2 \leqslant \gamma\delta_0$, does not leave the region $L(s) \leqslant c_1$ during the semi-interval $[\tau_i, \tau_{i+1})$, i.e. the fulfillment of the inequality $L(s) \leqslant c_1$ is ensured for $t \in [\tau_i, \tau_{i+1})$; whence follows the inequality $\|s_\Delta[t]\| \leqslant c_2$ or $\|x_\Delta[t] - w_\Delta[t]\| \leqslant \varepsilon_0 - \varepsilon$.

Thus, for $t \in [\tau_i, \tau_{i+1})$ we have $\rho(\{t, w_\Delta[t]\}, M) \geqslant \varepsilon_0$ and $\rho(\{t, x_\Delta[t]\}, \{t, w_\Delta \times [t]\}) \geqslant \varepsilon_0 - \varepsilon$. Here $\rho(\{t, w_\Delta[t]\}, M)$ is the distance from the point $\{t, w_\Delta[t]\}$ to set M in the Euclidean metric. Then

$$\rho(\{t, x_\Delta[t]\}, M) \geqslant |\rho(\{t, w_\Delta[t]\}, M) - \rho(\{t, x_\Delta[t]\}, \{t, w_\Delta[t]\})| \geqslant \varepsilon$$

The result obtained can be formulated as a theorem.

Theorem. Suppose that the following conditions are fulfilled for the initial position $\{t_0, x_0\}$:

1) whatever be the instant $\vartheta \in [t_0, \infty)$ and the strategy $U_* \div u_*(t, w, v_*)$, at least one motion $w[t, t_0, w_0, U_*]$ remains in H for $t \in [t_0, \vartheta]$;

2) system (1.4) is stabilizable.

Then we can find a strategy $V \div \{v(\tau, x, w), u_*(\tau, x, w), v_*(t, \tau, x, w, u_*(\cdot))\}$ of the control with leader such that for arbitrarily small $\alpha > 0$ and $\varepsilon > 0$ $(\varepsilon < \varepsilon_0)$, we can find a number $\delta_0 > 0$ such that evasion from the ε-neighborhood M^ε of set M is ensured during the infinite time semi-interval for all motions $x_\Delta[t] = x_\Delta[t, t_0, x_0, V, u(\cdot)]$ generated by this strategy and having the step $\sup_i(\tau_{i+1} - \tau_i) \leqslant \delta_0$ $(i = 0, 1, ..)$

In conclusion we note that a complete description of bridge $W_{\varepsilon_0}^\infty$ is, in general, not required when constructing the control V in concrete cases, but it is sufficient to know only how to compute, for each selected control u_*, the control v_* which retains the motion $w_\Delta[t]$ on the bridge $W_{\varepsilon_0}^\infty$ for $\tau_i \leqslant t < \tau_{i+1}$. Thus, the proposed stable procedure of position control V of the actual system (1.1) can be applied right away in any case when for the model (1.2) we know or we can effectively find the solution of the ε-evasion problem under information discrimination. Sometimes this can lead to a very simply realizable procedure of position control.

For example, we examine the evasion problem for a pair of objects of the same type [7], where the condition of contact is the coincidence of vectors y and z

$$dy / dt = Ay + Bu, \qquad u \in P \tag{2.13}$$
$$dz / dt = Az + Bv, \qquad v \in Q_*$$

Assume that among the eigenvalues of matrix A there is at least one with a positive real part; the system

$$ds / dt = As + Bm \tag{2.14}$$

is stabilizable and we can choose P_* so as to fulfill the conditions

$$P_* \supset P^\alpha, \quad Q_* \supset Q^\alpha, \quad Q = \varkappa P_* \quad (0 < \varkappa < 1)$$

We set $x = y - z$ and we write the model's equation as

$$dw \,/\, dt = Aw + Bu_* - Bv_*, \quad u_* \in P_*, \quad v_* \in Q$$

If the initial position $\{t_0, y_0, z_0\}$ is such that it is impossible to bring system (2.14) into the ε-neighborhood of point $s = 0$ in finite time by a choice of control $m \in (1 - \varkappa)\ P_*$, then to retain the position $\{t, w[t]\}$ on bridge $W_{\varepsilon_0}^\infty$ it is sufficient to choose v_* such that $u_* - v_* \in (1 - \varkappa)\ P_*$. Thus, in the given example all the needed constructions connected with the bridge $W_{\varepsilon_\iota}^\infty$ turn out to be very simple, although the description of the bridge itself remains unknown.

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QUALITATIVE INVESTIGATION OF A PIECEWISE LINEAR SYSTEM

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We use the methods of the theory of bifurcation and piecewise linear approximation to the characteristic with a falling segment, in the qualitative investigation of a system which is of practical interest. We trace the possible bifurcations and follow the behavior of the bifurcation curves. The system has been studied by a number of authors, using various approximations [1 – 9], however none of them gave a complete qualitative investigation.

1. Equations of motion. We consider the system